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Transfinite almost square Banach spaces

by

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Abstract. Is is known that a Banach space contains an isomorphic copy of c_0 if, and only if, it can be equivalently renormed to be almost square. We introduce and study transfinite versions of almost square Banach spaces with the purpose of relating them to the containment of isomorphic copies of $c_0(\kappa)$, where κ is some uncountable cardinal. We also provide several examples and stability results for the above properties by taking direct sums, tensor products and ultraproducts. By connecting the above properties with transfinite analogues of the strong diameter 2 property and octahedral norms, we obtain a solution to an open question of Ciaci et al. [Israel J. Math. (online, 2022)].

1. Introduction. Since the starting point of the study of Banach space theory, a considerable effort has been made in order to determine how the presence of an isomorphic copy of c_0 or ℓ_1 in a Banach space affects its structure. This makes interesting the search of properties which characterise the containment of the above-mentioned spaces. In this sense, let us indicate two characterisations of the containment of the spaces ℓ_1 and c_0 of geometric nature. In [14, Theorem II.4] it is proved that a Banach space X contains an isomorphic copy of ℓ_1 if, and only if, it admits an equivalent norm $|||\cdot|||$ which is *octahedral*, that is, given a finite-dimensional subspace $Y \subset X$ and $\varepsilon > 0$, there is an element $x \in S_{(X,|||\cdot|||)}$ such that

$$|||y + rx||| \geq (2-\varepsilon)(|||y||| + |r|) \quad \text{for all } y \in Y \text{ and } r \in \mathbb{R}.$$

Concerning the containment of c_0, a more recent characterisation was given in [8, Corollary 2.4]: a Banach space X contains an isomorphic copy of c_0 if, and only if, it admits an equivalent *almost square* (*ASQ*, for short) norm $|||\cdot|||$, that is, given a finite-dimensional subspace $Y \subset X$ and $\varepsilon > 0$, there is an

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element $x \in S_{(X,|||\cdot|||)}$ such that

$$|||y + rx||| \leq (1+\varepsilon)(|||y||| \vee |r|) \quad \text{for all } y \in Y \text{ and } r \in \mathbb{R}.$$

At this point, it is natural to look for geometric characterisations of the containment of non-separable versions of ℓ_1 and c_0. In this spirit, as far as the containment of $\ell_1(\kappa)$ is concerned, transfinite generalisations of octahedral norms were introduced in [9] in various directions and some characterisations of the containment of $\ell_1(\kappa)$ were obtained in [6, 9]. To mention the strongest known result, it is proved in [6, Theorem 1.3] that a Banach space X contains an isomorphic copy of $\ell_1(\kappa)$, where κ is an uncountable cardinal, if, and only if, there exists an equivalent norm $|||\cdot|||$ such that $(X, |||\cdot|||)$ fails the (-1)-*ball covering property for cardinals* $< \kappa$ ((-1)-*BCP*$_{<\kappa}$, for short), which means that, given any subspace $Y \subset X$ such that $\text{dens}(Y) < \kappa$, there exists $x \in S_{(X,|||\cdot|||)}$ such that

$$|||y + rx||| = |||y||| + |r| \quad \text{for all } y \in Y \text{ and } r \in \mathbb{R}.$$

Motivated by the above results, in the present paper we aim to introduce different transfinite versions of ASQ spaces in order to search for a characterisation of those spaces that contain isomorphic copies of $c_0(\kappa)$.

Let us now describe in more detail the content of the paper. In Section 2 we define transfinite ASQ spaces and call them ASQ$_{<\kappa}$ spaces, where κ is some fixed cardinal (see Definition 2.1(a)), and we provide many examples of Banach spaces enjoying these properties. In Section 3 we consider the relations between being ASQ$_{<\kappa}$, admitting an equivalent ASQ$_{<\kappa}$ renorming and containing isomorphic copies of $c_0(\kappa)$. One of the highlights of this section is Example 3.1, in which we find, for every uncountable cardinal κ, a Banach space X which is ASQ$_{<\kappa}$, but such that X^* fails to contain $\ell_1(\omega_1)$ and, in particular, X cannot contain $c_0(\omega_1)$. This means that the property of being renormable to become ASQ$_{<\kappa}$ is not strong enough to characterise the Banach spaces that contain $c_0(\kappa)$ isomorphically. Hence, we consider a strengthening of ASQ$_{<\kappa}$ that we call SQ$_{<\kappa}$ spaces (see Definition 2.1(b)) and which will contain isomorphic copies of $c_0(\kappa)$. Therefore we face the question whether every Banach space containing $c_0(\kappa)$, for some uncountable cardinal κ, admits an equivalent SQ$_{<\kappa}$ renorming. Even though we do not know the answer in complete generality, we prove in Theorem 3.7 that, if $\text{dens}(X) = \kappa$, then X admits an equivalent SQ$_{<\text{cf}(\kappa)}$ renorming, where $\text{cf}(\kappa)$ stands for the cofinality of κ.

In Section 4 we study various stability results for (A)SQ$_{<\kappa}$ spaces under different operations on Banach spaces, in order to enlarge the class of the known examples of Banach spaces enjoying these properties. We mainly extend known results to ASQ spaces, but which, in their transfinite version, produce new surprising results. For instance, with respect to absolute sums, in Theorem 4.1 we are able to produce ℓ_∞-sums of spaces which are $\mathrm{ASQ}_{<\kappa}$ even though none of their components is $\mathrm{ASQ}_{<\kappa}$, which is a notable difference from the previously known results for finite sums of ASQ spaces. We also analyse (A)$\mathrm{SQ}_{<\kappa}$ properties with respect to taking spaces of operators and tensor products. In Corollary 4.7 we prove that, if X is (A)$\mathrm{SQ}_{<\kappa}$ and Y is non-trivial, then the injective tensor product $X \widehat{\otimes}_\varepsilon Y$ is (A)$\mathrm{SQ}_{<\kappa}$. If we also require Y being (A)$\mathrm{SQ}_{<\kappa}$, then so is its projective tensor product $X \widehat{\otimes}_\pi Y$ (see Proposition 4.4). Observe that the latter result is important, because most of the known examples of ASQ spaces come from some kind of ∞-norm, but the projective norm on a tensor product has dramatically different behaviour.

We end the study of stability results with ultraproducts, which, as one might expect, provide a lot of examples of $\mathrm{SQ}_{<\kappa}$ spaces. Indeed, in Proposition 4.8 we prove that, if a family $\{X_\alpha : \alpha \in \mathcal{A}\}$ consists of $\mathrm{ASQ}_{<\kappa}$ spaces and if we consider a countable incomplete ultrafilter $\mathcal{U}$, then the ultraproduct $(X_\alpha)_\mathcal{U}$ is $\mathrm{SQ}_{<\kappa}$. Lastly, we show in Example 4.9 that the requirement of the factors being $\mathrm{ASQ}_{<\kappa}$ is not necessary.

In Section 5 we investigate the connection of (A)$\mathrm{SQ}_{<\kappa}$ properties with other properties of Banach spaces, such as the transfinite versions of octahedrality and diameter 2 properties. From our work we derive that if X is $\mathrm{ASQ}_{<\kappa}$ (respectively, $\mathrm{SQ}_{<\kappa}$), then X has the $\mathrm{SD2P}_{<\kappa}$ (respectively, the 1-$\mathrm{ASD2P}_{<\kappa}$), and consequently, X^* is $<\kappa$-octahedral (respectively, fails the (-1)-$\mathrm{BCP}_{<\kappa}$) (see Proposition 5.4). As a consequence, in Remark 5.5 we provide, for every uncountable cardinal κ, an example of a Banach space X which is $<\kappa$-octahedral but which fails to contain an isomorphic copy of $\ell_1(\omega_1)$, giving a negative solution to [9, Question 1].

In Section 6 we introduce a parametric version of (A)$\mathrm{SQ}_{<\kappa}$ spaces (see Definition 6.1) which includes all of the known versions of ASQ spaces. Moreover, we improve the known isomorphic characterisation of Banach spaces containing c_0 (see Theorem 6.8).

Terminology. Throughout the paper we only consider real Banach spaces. Given a Banach space X, we denote the closed unit ball and the unit sphere of X by B_X and S_X, respectively. We denote by X^* the topological dual of X. Given two Banach spaces X and Y we denote by $L(X,Y)$ the space of linear bounded operators from X into Y. Given a subset A of X we denote by $\mathrm{span}(A)$ (respectively, $\overline{\mathrm{span}}(A)$) the linear span (respectively, the closed linear span) of A, whereas $\mathrm{dens}(X)$ denotes the *density* character of a topological space X, i.e. the smallest cardinality of a dense set in X.

Given a set A, we denote by $|A|$ its cardinality and by $\mathcal{P}_\kappa(A)$ and $\mathcal{P}_{<\kappa}(A)$ the sets of all subsets of A of cardinality at most κ and strictly less than κ, respectively, for some cardinal κ. We denote by $\mathbb{N}_{\geq 2}$ the set $\{n \in \mathbb{N} : n \geq 2\}$.

Given a cardinal κ, we denote by $\mathrm{cf}(\kappa)$ its *cofinality*, which is the smallest cardinal λ such that κ can be expressed as a union of λ many sets of cardinality strictly smaller than κ. We use $\mathfrak{c}$ to denote the cardinality of the continuum.

Given an infinite set $\mathcal{A}$ and an uncountable cardinal κ, a non-principal ultrafilter $\mathcal{U}$ over $\mathcal{A}$ is said to be *κ-complete* if it closed with respect to $< \kappa$ many intersections. It is immediate to see that a non-principal ultrafilter $\mathcal{U}$ is $\aleph_1$-incomplete if, and only if, there is a function $f : \mathcal{A} \to \mathbb{R}$ such that $f(\alpha) > 0$ for every $\alpha \in \mathcal{A}$ and $\lim_{\mathcal{U}} f(\alpha) = 0$.

We refer the reader to [20] for background on set theory and cardinals.

We use the standard notation for lattices: $x \vee y$ denotes the supremum of elements x and y and $\bigvee A$ denotes the supremum of a set A.

2. Transfinite almost square Banach spaces and first examples.

Let us begin with the definition of an (A)SQ Banach space depending on a given cardinal κ.

DEFINITION 2.1. Let X be a Banach space and κ a cardinal.

(a) We say that X is *$<\kappa$-almost square* (*$ASQ_{<\kappa}$*, for short) if, for every set $A \in \mathcal{P}_{<\kappa}(S_X)$ and $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$ for all $x \in A$,

(b) We say that X is *$<\kappa$-square* (*$SQ_{<\kappa}$*, for short) if, for every set $A \in \mathcal{P}_{<\kappa}(S_X)$, there exists $y \in S_X$ such that $\|x \pm y\| \leq 1$ for all $x \in A$.

As a special case, let us also define ASQ_κ and SQ_κ spaces by considering $A \in \mathcal{P}_\kappa(S_X)$ instead.

Notice that, if κ is infinite, then we can equivalently require just that $\|x + y\| \leq 1 + \varepsilon$. Moreover, a standard argument shows that for the SQ case it is equivalent to require that $\|x \pm y\| = 1$.

It is known that a Banach space X is ASQ if, and only if, for every finite-dimensional subspace $Y \subset X$ and $\varepsilon > 0$, there is $x \in S_X$ such that

$$\|y + rx\| \leq (1 + \varepsilon)(\|y\| \vee |r|) \quad \text{for all } y \in Y \text{ and } r \in \mathbb{R}$$

(see [2, Proposition 2.1]). Even more is true: a straightforward application of [2, Lemma 2.2] provides a description of the above notions via subspaces of density $< \kappa$ whenever κ is uncountable. The version for $\kappa = \aleph_0$ is unknown to the authors.

PROPOSITION 2.2. *Let X be a Banach space and κ an uncountable cardinal. The following are equivalent:*

(i) *X is $ASQ_{<\kappa}$ (respectively, $SQ_{<\kappa}$).*

(ii) *For every subspace* $Y \subset X$ *with* $\operatorname{dens}(Y) < \kappa$ *and* $\varepsilon > 0$ (*respectively,* $\varepsilon \geq 0$), *there exists* $x \in S_X$ *such that*

$$\|y + rx\| \leq (1+\varepsilon)(\|y\| \vee |r|) \quad \textit{for every } y \in Y \textit{ and every } r \in \mathbb{R}.$$

We devote the rest of the section to various examples of transfinite (A)SQ spaces.

Example 2.3. Let κ be an uncountable cardinal, and let $\ell_\infty^c(\kappa)$ be the elements of $\ell_\infty(\kappa)$ whose support is at most countable. If X is a subspace of $\ell_\infty^c(\kappa)$ containing $c_0(\kappa)$, then X is $\mathrm{SQ}_{<\kappa}$. Indeed, fix $A \in \mathcal{P}_{<\kappa}(S_X)$. Since $\operatorname{supp}(f) \subseteq \kappa$ is at most countable for every $f \in A$, we can find $\lambda \in \kappa$ such that $\lambda \notin \bigcup_{f\in A} \operatorname{supp}(f)$. Clearly $\|f + e_\lambda\| = 1$ for every $f \in A$.

Example 2.4. Fix a non-principal ultrafilter $\mathcal{U}$ in $\mathbb{N}$. For every $x \in \ell_\infty$, denote by $\lim(x)$ the limit of $x(n)$ with respect to $\mathcal{U}$ and define the norm

$$|||x||| := |\lim(x)| \vee \bigvee_{n\in\mathbb{N}} |x(n) - \lim(x)|.$$

The Banach space $X := (\ell_\infty, |||\cdot|||)$ was defined in [8] and proved to be ASQ. In the following we prove that it actually is $\mathrm{SQ}_{<\aleph_0}$. Nevertheless, X cannot be $\mathrm{ASQ}_{\aleph_0}$ (see Theorem 3.3) since $\ell_\infty = C(\beta\mathbb{N})$.

Fix $x_1, \ldots, x_k \in S_X$ and define, for every $n \in \mathbb{N}$ and $m \in \{1, \ldots, k\}$,

$$A_{n,m} := \{p \in \mathbb{N} : |x_m(p) - \lim(x_m)| < 1/n\}.$$

By the definition of ultralimit, $A_{n,m} \in \mathcal{U}$, therefore $A_n := \bigcap_{m=1}^k A_{n,m} \in \mathcal{U}$ for every $n \in \mathbb{N}$. Since $\mathcal{U}$ is non-principal, each A_n is infinite, hence, for every $n \in \mathbb{N}$, we can find $f(n) \in A_n$ such that $f(n) < f(n+1)$. Notice that, since $\emptyset \notin \mathcal{U}$, either $f(2\mathbb{N})$ or $f(2\mathbb{N}+1)$ is not in $\mathcal{U}$, say $f(2\mathbb{N}) \notin \mathcal{U}$. Define the formal series

$$y := \sum_{n\in 2\mathbb{N}} (1 - 1/n) e_{f(n)} \in \ell_\infty.$$

Notice that $\|y\|_\infty = 1$ and $\lim(y) = 0$, so $y \in S_X$. For every $i \in \{1, \ldots, k\}$, notice that

$$\begin{aligned} |||x_i + y||| &= |\lim(x_i)| \vee \bigvee_{n\in\mathbb{N}} |x_i(n) - \lim(x_i) + y(n)| \\ &\leq 1 \vee \bigvee_{n\in 2\mathbb{N}} |x_i(f(n)) - \lim(x_i) + (1 - 1/n)| \vee \bigvee_{n\in\mathbb{N}\setminus f(2\mathbb{N})} |x_i(n) - \lim(x_i)| \\ &\leq 1 \vee (1 - 1/n + 1/n) \vee 1 = 1. \end{aligned}$$

Therefore X is $\mathrm{SQ}_{<\aleph_0}$.

The previous example yields a non-separable example of an $\mathrm{SQ}_{<\aleph_0}$ space. A natural question at this point is whether or not there is a separable Banach space which is $\mathrm{SQ}_{<\aleph_0}$. The next example is a modification of [2, Example 6.4] and provides an affirmative answer.

Example 2.5. Given $n \in \mathbb{N}$, consider

$$X_n := \{f \in C(S_{\mathbb{R}^n}) : f(s) = -f(-s) \text{ for all } s \in S_{\mathbb{R}^n}\}.$$

Let us show that X_n is SQ_n. Fix $f_1, \ldots, f_n \in S_{X_n}$. By a corollary of the Borsuk–Ulam theorem [4, p. 485, Satz VIII], we can find $s_0 \in S_{\mathbb{R}^n}$ such that $f_i(s_0) = 0$ for every $i \in \{1, \ldots, n\}$. Pick any function $h \in S_{X_n}$ such that $h(s_0) = 1$ and define

$$g(s) := \Big(1 - \bigvee_{i=1}^{n} |f_i(s)|\Big) h(s).$$

Notice that $g \in X_n$ and that $g(s_0) = 1$, therefore $\|g\| = 1$. For every $i \in \{1, \ldots, n\}$ and $s \in S_{\mathbb{R}^n}$ we have

$$|f_i(s) \pm g(s)| \le |f_i(s)| + |g(s)| \le |f_i(s)| + 1 - \bigvee_{j=1}^{n} |f_j(s)| \le 1,$$

as required.

Now define $X := c_0(\mathbb{N}, X_n)$. It is obvious that X is separable. Moreover, since X_n is SQ_n for every $n \in \mathbb{N}$, it is immediate to check that X is $\mathrm{SQ}_{<\aleph_0}$. In fact, fix $x_1, \ldots, x_k \in S_X$ and without loss of generality assume that $\|x_i(k)\| = 1$ for all $i \in \{1, \ldots, k\}$. Find $y \in S_{X_k}$ such that $\|x_i(k) + y\| \le 1$ for all $i \in \{1, \ldots, k\}$, therefore $\|x_i + y \cdot e_k\| \le 1$.

In [15], spaces of (almost) universal disposition were introduced. Let us recall their definitions. Given a family $\mathfrak{K}$ of Banach spaces, a Banach space X is *of almost universal disposition for* $\mathfrak{K}$ if, for every $S \subset T$ in $\mathfrak{K}$, any isometric embedding $f : S \to X$ extends to an ε-isometric embedding $F : T \to X$. Moreover, a Banach space X is *of universal disposition for* $\mathfrak{K}$ if for every $S \subset T$ in $\mathfrak{K}$, any isometric embedding $f : S \to X$ extends to an isometric embedding $F : T \to X$.

The Gurariĭ space is the classical example of a space of almost universal disposition for finite-dimensional spaces. We refer to [5] for more examples and further discussion of this kind of properties.

Example 2.6. If X is of almost universal disposition (respectively, of universal disposition) for Banach spaces with density character strictly less then κ, then X is $\mathrm{ASQ}_{<\kappa}$ (respectively, $\mathrm{SQ}_{<\kappa}$). We show the claim for the ASQ case only. Fix a subspace $Y \subset X$ with $\mathrm{dens}(Y) < \kappa$ and $\varepsilon > 0$. The inclusion $Y \to X$ extends to an ε-isometrical embedding $T : Y \oplus_\infty \mathbb{R} \to X$. Find $r \in \mathbb{R}$ such that $\|T(0, r)\| = 1$. We can do so since T is injective and, by picking any $s \neq 0$, we can set $r := s/T(0, s)$. Notice that

$$|r| = \|(0, r)\|_\infty \le (1 + \varepsilon)\|T(0, r)\| = 1 + \varepsilon.$$

It is clear that, for every $y \in S_Y$,

$$\|y + T(0,r)\| \leq (1+\varepsilon)\|(y,0) + (0,r)\|_\infty = (1+\varepsilon)(\|y\| \vee |r|) \leq (1+\varepsilon)^2.$$

In the following we study $C_0(X)$ spaces. It is known that, given a locally compact Hausdorff space X, $C_0(X)$ is ASQ if, and only if, X is non-compact [7, Proposition 2.1]. Below we provide a topological description of X such that $C_0(X)$ is ASQ$_{<\kappa}$ whenever κ is uncountable, and as a byproduct, we find that being SQ$_{<\kappa}$ and ASQ$_{<\kappa}$ are equivalent in $C_0(X)$ spaces, at least under a mild regularity assumption on X.

THEOREM 2.7. *Let X be a T_4 locally compact space. If κ is an uncountable cardinal, then the following are equivalent:*

(i) *$C_0(X)$ is SQ$_{<\kappa}$,*
(ii) *$C_0(X)$ is ASQ$_{<\kappa}$,*
(iii) *if $\mathcal{K} \in \mathcal{P}_{<\kappa}(\mathcal{P}(X))$ is a family consisting of compact sets in X, then $\bigcup \mathcal{K}$ is not dense in X.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Fix a family $\mathcal{K} \in \mathcal{P}_{<\kappa}(\mathcal{P}(X))$ consisting of compact sets in X and fix any $K \in \mathcal{K}$. Since K is compact and X is locally compact, we can find a covering $U_1, \ldots, U_n$ for K consisting of open relatively compact sets. Define $U := \bigcup_{i=1}^n U_i$ and notice that $X \setminus U \neq \emptyset$, otherwise we would get $X = \overline{U}$, which is compact, and this would contradict the fact that $C_0(X)$ is ASQ$_{<\kappa}$. On the other hand, it is clear that K and $X \setminus U$ are disjoint closed sets, therefore, since X is normal, there exists an Urysohn function $f_K : X \to [0,1]$ such that $f_K|_K = 1$ and $f_K|_{X\setminus U} = 0$. Notice that the support of f_K is contained in $\overline{U}$, which is compact, thus $f_K \in S_{C_0(X)}$. Since $C_0(X)$ is ASQ$_{<\kappa}$, there is $g \in S_{C_0(X)}$ satisfying

$$\|f_K \pm g\|_\infty \leq 3/2 \quad \text{for every } K \in \mathcal{K}.$$

It is clear by construction that $|g(x)| \leq 1/2$ for every $x \in \bigcup \mathcal{K}$. Therefore the non-empty open set $\{x \in X : |g(x)| > 1/2\}$ is disjoint from $\bigcup \mathcal{K}$, hence $\bigcup \mathcal{K}$ is not dense in X.

(iii)⇒(i). Fix $A \in \mathcal{P}_{<\kappa}(S_{C_0(X)})$. For every $f \in A$ and $n \in \mathbb{N}$, there exists a compact set $K_{f,n} \subset X$ such that $|f(x)| < 1/n$ for every $x \in X \setminus K_{f,n}$. Define $\mathcal{K} := \{K_{f,n} : f \in A \text{ and } n \in \mathbb{N}\}$ and notice that $|\mathcal{K}| \leq |A| \cdot \aleph_0 < \kappa$ since κ is uncountable. By assumption we can find a non-empty open set U which is disjoint from $\bigcup \mathcal{K}$ and, without loss of generality, we can assume it is relatively compact. Since X is normal, there exists an Urysohn function $g : X \to [0,1]$ such that $\|g\|_\infty = 1$ and $g|_{X\setminus U} = 0$. Notice that the support of g is contained in $\overline{U}$, which is compact, thus $g \in S_{C_0(X)}$. It is clear by construction that $\|f + g\|_\infty = 1$ for every $f \in A$. ∎

A closer look at the proof of Theorem 2.7 reveals that (ii)⇔(iii) actually holds without the assumption that κ is uncountable. This corresponds to the already recalled result that $C_0(X)$ is ASQ if, and only if, X is non-compact.

Let us note that in the case $\kappa = \aleph_1$, property (iii) of Theorem 2.7 can be stated as "X does not admit a dense sigma-compact set".

3. Banach spaces which admit transfinite ASQ renorming. Let X be a Banach space. In this section we will analyse the relations between the following properties:

(a) X admits an equivalent $\mathrm{ASQ}_{<\kappa}$ renorming,
(b) X admits an equivalent $\mathrm{SQ}_{<\kappa}$ renorming,
(c) X contains an isomorphic copy of $c_0(\kappa)$.

Observe that an easy transfinite induction argument reveals that, if X admits an equivalent $\mathrm{SQ}_{<\kappa}$ renorming, then X contains an isomorphic copy of $c_0(\kappa)$. The situation is dramatically different if we replace the SQ norm with an ASQ norm, as we can see in the next example.

We denote

$$\ell_p(\kappa) := \Big\{x : \kappa \to \mathbb{R} : \|x\| := \Big(\sum_{\eta<\kappa} |x(\eta)|^p\Big)^{1/p} < \infty\Big\}$$

and

$$c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa)) := \Big\{x : \mathbb{N}_{n\geq 2} \to \prod_n \ell_n(\kappa) : x(n) \in \ell_n(\kappa) \text{ and } \lim_n \|x(n)\| = 0\Big\}$$

endowed with the norm $\|x\| = \bigvee_n \|x(n)\|$.

Example 3.1. Let κ be an infinite cardinal. Then $X := c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ is $\mathrm{ASQ}_{<\kappa}$ but X^* does not contain any isomorphic copy of $\ell_1(\omega_1)$; in particular, X does not contain any isomorphic copy of $c_0(\omega_1)$.

Proof. Let us initially suppose that $\kappa > \aleph_0$. In order to prove that X is $\mathrm{ASQ}_{<\kappa}$, it is enough to note that, for every set $A \in \mathcal{P}_{<\kappa}(S_{\ell_n(\kappa)})$, there is $y \in S_{\ell_n(\kappa)}$ such that $\|x + y\| \leq 2^{1/n}$ for all $x \in A$; call this property $(2^{-1/n}, 2^{-1/n})$-$\mathrm{SQ}_{<\kappa}$. Indeed, since every $x \in A$ has countable support, we can find $\lambda \in \kappa$ such that $x(\lambda) = 0$ for all $x \in A$. Take y defined by $y(\mu) = \delta_{\lambda\mu}$. Now $\|x + y\| = (\|x\|^n + \|y\|^n)^{1/n} = 2^{1/n}$ for all $x \in A$, as required. To conclude, fix $A \in \mathcal{P}_{<\kappa}(S_X)$ and $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $2^{1/n} < 1 + \varepsilon$ and find $y \in S_{\ell_n(\kappa)}$ such that $\|x(n)+y\| \leq 2^{1/n}$ for all $x \in A$; then $\|x+y\cdot e_n\| \leq 1 + \varepsilon$.

If $\kappa = \aleph_0$, the proof is similar, but, for every $\varepsilon > 0$, we can only manage to find y such that $\|x + y\| \leq (2 + \varepsilon)^{1/n}$, which is still enough. Indeed, given a finite set $A \subset S_{\ell_n}$, we can find $m \in \mathbb{N}$ such that $|x(m)| < \delta$ for all $x \in A$, where $\delta > 0$ is chosen such that $(1+\delta)^n < 1+\varepsilon$, thus we only need to define $y := e_m$.

In order to prove the second part, observe that $X^* = \ell_1(\mathbb{N}_{\geq 2}, \ell_{n^*}(\kappa))$ where n^* is the conjugate index of n. Since X^* is a countable sum of reflexive Banach spaces, we deduce that X^* is weakly compactly generated. Consequently, X^* cannot contain $\ell_1(\omega_1)$, which even fails to have weaker properties (e.g., by using [11, Theorem 12.42] it is immediate to see that $\ell_1(\omega_1)$ fails the Corson property (C), which is inherited by closed subspaces).

Finally, to conclude that X does not contain $c_0(\omega_1)$, observe that if X contained $c_0(\omega_1)$, then taking adjoints we would see that $\ell_1(\omega_1)$ would be isomorphic to a quotient of X^*. Since $\ell_1(\omega_1)$ has the lifting property, we would conclude that $\ell_1(\omega_1)$ is isomorphic to a subspace of X^*, which entails a contradiction with the previous point. ■

Remark 3.2. The same proof as in Example 3.1 shows that $\ell_\infty(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ is also $\mathrm{ASQ}_{<\kappa}$. In more detail, using the terminology from Example 3.1, since each $\ell_n(\kappa)$ is $(2^{-1/n}, 2^{-1/n})$-$\mathrm{SQ}_{<\kappa}$ (see Definition 6.1), the claim follows from a direct computation or from Theorem 4.1. This proves that, for every infinite cardinal κ, there are dual (actually bidual) Banach spaces which are $\mathrm{ASQ}_{<\kappa}$. Let us point out that the importance of this result is that, for classical ASQ spaces, it was asked in [2] whether there is any dual ASQ space, which was answered in the affirmative in [1].

Observe that the situation for SQ spaces is different, because they are clearly incompatible with the existence of extreme points in the unit ball, so no dual Banach space can enjoy any SQ property.

We have seen that the $\mathrm{ASQ}_{<\kappa}$ condition does not imply the containment of large copies of c_0. However, this behaviour is impossible in spaces of continuous functions, as the following theorem shows.

Theorem 3.3. *Let K be a compact Hausdorff topological space. If $C(K)$ admits any equivalent $ASQ_{<\kappa}$ norm, then it contains an isomorphic copy of $c_0(\kappa)$.*

Proof. Set $X := C(K)$ and assume that

$$\frac{1}{M}\|f\| \leq |||f||| \leq M\|f\| \quad \text{for every } f \in X.$$

Find $p \in \mathbb{N}$ large enough and $\varepsilon > 0$ small enough so that $p > 2M^2(1+\varepsilon)^p$. If $(X, |||\cdot|||)$ is $\mathrm{ASQ}_{<\kappa}$ then, by transfinite induction, we can find $\{f_\alpha : \alpha < \kappa\} \subseteq S_{(X,|||\cdot|||)}$ such that

$$|||f + rf_\alpha||| \leq (1+\varepsilon)(|||f||| \vee |r|) \quad \text{for every } f \in \overline{\mathrm{span}}\,\{f_\beta : \beta < \alpha\} \text{ and } r \in \mathbb{R}.$$

Note that $\|f_\alpha\| \geq 1/M$. Up to considering $-f_\alpha$ instead, we can assume that the set

$$V_\alpha := \left\{ x \in K : f_\alpha(x) > \frac{1}{2M} \right\}$$

is non-empty, and it is clearly open. Suppose for contradiction that $c_0(\kappa)$ does not embed in $C(K)$; we then infer that K satisfies the κ-chain condition [25, p. 227]. By [25, p. 227, Remark], since $\{V_\alpha : \alpha < \kappa\}$ is a family of open sets in K, we can find an infinite set $\{\alpha_n : n \in \mathbb{N}\} \subseteq [0, \kappa)$ such that there exists $x \in \bigcap_{n\in\mathbb{N}} V_{\alpha_n}$. Eventually,

$$(1+\varepsilon)^p > \left\| \sum_{i=1}^{p} f_i \right\| \geq \frac{1}{M} \left\| \sum_{i=1}^{p} f_i \right\| \geq \frac{1}{M} \sum_{i=1}^{p} f_i(x) \geq \frac{p}{2M^2},$$

which is a contradiction. ∎

Now it is time to analyse the following question.

Problem 3.4. *Let κ be an infinite cardinal. If a Banach space contains an isomorphic copy of $c_0(\kappa)$, does it admit an equivalent $SQ_{<\kappa}$ renorming?*

We do not know the answer to the above question in complete generality. However, we are able to give some partial positive answers. The first one deals with Banach spaces that contain c_0.

Proposition 3.5. *If X is a dual Banach space containing an isomorphic copy of c_0, then X admits an equivalent $SQ_{<\aleph_0}$ renorming.*

Proof. If X is a dual Banach space containing c_0, then X contains an isomorphic copy of ℓ_∞ [23, Proposition 2.e.8]. Because of its injectivity, ℓ_∞ is complemented in X (cf. e.g. [11, Proposition 5.13]). Consequently, there is a subspace Z of X such that $X = \ell_\infty \oplus Z$. Consider the norm $|||\cdot|||$ on ℓ_∞ described in Example 2.4. Now, consider on X the equivalent norm such that $X = (\ell_\infty, |||\cdot|||) \oplus_\infty Z$. Then X, endowed with this norm, is $\mathrm{SQ}_{<\aleph_0}$, because $(\ell_\infty, |||\cdot|||)$ is $\mathrm{SQ}_{<\aleph_0}$ by applying Corollary 4.2 below. ∎

We do not know whether c_0 has an equivalent $\mathrm{SQ}_{<\aleph_0}$ renorming and, when $\kappa > \aleph_0$, we do not know if $\ell_\infty(\kappa)$ has an equivalent $\mathrm{SQ}_{<\kappa}$ renorming. The best we can say in this direction is the following.

Proposition 3.6. *Let κ and λ be uncountable cardinals. If there exists a κ-complete ultrafilter $\mathcal{U}$ on λ, then $\ell_\infty(\lambda)$ admits an equivalent $SQ_{<\kappa}$ renorming.*

Proof. Define an equivalent norm by the same formula as in Example 2.4:

$$|||x||| := |\lim(x)| \vee \bigvee_{\mu\in\lambda} |x(\mu) - \lim(x)|,$$

where $\lim(x)$ denotes the limit through the ultrafilter $\mathcal{U}$. As before, if we take $X \in \mathcal{P}_{<\kappa}(S_{\ell_\infty(\lambda)})$, then

$$A_{n,x} = \{\mu \in \lambda : |x(\mu) - \lim(x)| < 1/n\} \in \mathcal{U} \quad \text{for all } n \in \mathbb{N} \text{ and } x \in X.$$

So $A := \{\mu \in \lambda : x(\mu) = \lim(x)\} = \bigcap_{n,x} A_{n,x} \in \mathcal{U}$. If we take $\mu \in A$, then it is easily checked that $\|\!|x + e_\mu|\!\| = 1$ for all $x \in X$. ∎

This statement is quite unsatisfactory because λ must be a large cardinal, at least the first measurable cardinal. Using a variation of this idea by taking multiple ultrafilters instead of just a fixed one, we obtain another general result which says that, when X contains $c_0(\kappa)$ and when $X/c_0(\kappa)$ is small in a sense, then X admits an equivalent $\mathrm{SQ}_{<\kappa}$ norm. This is the main result of this section.

THEOREM 3.7. *Let κ be an infinite cardinal of uncountable cofinality. If a Banach space of density character κ contains an isomorphic copy of $c_0(\kappa)$, then it admits an equivalent $SQ_{<\mathrm{cf}(\kappa)}$ renorming.*

Proof. Without loss of generality we can suppose that the copy of $c_0(\kappa)$ is isometric. Let $Y \subset X$ be a subspace together with an isometric isomorphism $S : Y \to c_0(\kappa)$. By Hahn–Banach, there exists a norm-1 operator $T : X \to \ell_\infty(\kappa)$ such that $T|_Y = S$.

Now we aim to define a suitable one-to-one mapping $g : \kappa \to B_{X^*}$ such that all $g(\alpha)$'s vanish on Y. After doing so, we define the equivalent norm

$$\|\!|x|\!\| := \|x\|_{X/Y} \vee \bigvee_{\alpha<\kappa} |T_\alpha(x) - g(\alpha)(x)|$$

First step: $\|\!| \cdot |\!\|$ defines an equivalent norm on X.

In fact, it is clear that $\|\!| \cdot |\!\|$ is a norm and that $\|\!| \cdot |\!\| \le 2\|\cdot\|$. Now suppose for contradiction that we cannot obtain the opposite inequality with respect to any fixed constant; then we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset S_X$ satisfying $\lim_n \|\!|x_n|\!\| = 0$. This implies that $\lim_n \|x_n\|_{X/Y} = 0$, so we can find elements $y_n \in Y$ such that $\lim_n \|x_n - y_n\| = 0$. This, as before, shows that $\lim_n \|\!|x_n - y_n|\!\| = 0$.

Since $\lim_n \|\!|x_n|\!\| = 0$, we conclude that $\lim_n \|\!|y_n|\!\| = 0$, but, as $y_n \in Y$, we get $\|\!|y_n|\!\| = \bigvee_{\alpha<\kappa} |T_\alpha(y_n)| = \|y_n\|$, hence $\lim_n \|y_n\| = 0$. This, together with $\lim_n \|x_n - y_n\| = 0$, implies that $\lim_n \|x_n\| = 0$, which is a contradiction.

Second step: If $T_\beta(x) = g(\beta)(x)$, then $\|\!|x + tS^{-1}(e_\beta)|\!\| = \|\!|x|\!\| \vee |t|$.

In fact, set $u_\beta := S^{-1}(e_\beta)$ and observe that

$$\|\!|x|\!\| = \|x\|_{X/Y} \vee \bigvee_{\alpha\neq\beta} |T_\alpha(x) - g(\alpha)(x)|.$$

By the fact that $g(\beta)(u_\beta) = 0$, we deduce that

$$|T_\beta(x + tu_\beta) - g(\beta)(x + tu_\beta)| = |t|.$$

Notice also that $T_\alpha(u_\beta) = g(\alpha)(u_\beta) = 0$. Therefore

$$\begin{aligned}\|\!|x+tu_\beta|\!\| &= \|x\|_{X/Y} \vee |T_\beta(x+tu_\beta) - g(\beta)(x+tu_\beta)| \vee \bigvee_{\alpha\neq\beta} |T_\alpha(x) - g(\alpha)(x)| \\ &= \|\!|x|\!\| \vee |t|.\end{aligned}$$

Third step: If $Z \subset X$ is a subspace with $\mathrm{dens}(Z) < \kappa$, then, for all $\alpha \in \kappa$, except for $< \kappa$ many α's, there exist functionals $g_\alpha \in B_{X^*}$ that vanish on Y and are such that $T_\alpha(x) = g_\alpha(x)$ for all $x \in Z$.

In fact, consider the continuous function $\phi : \beta\kappa \to B_{Z^*}$ given by

$$\phi(\mathcal{U})(z) = \lim_{\mathcal{U}} T_\gamma(z),$$

where the limit is taken with respect to γ, and the topology on B_{Z^*} is the weak* topology. Notice that, for $\alpha < \kappa$, if a non-principal ultrafilter $\mathcal{U}_\alpha$ in κ satisfies $\phi(\mathcal{U}_\alpha) = \phi(\alpha)$, then $g_\alpha := \lim_{\mathcal{U}_\alpha} T_\gamma$ satisfies the desired conditions. So it is enough to show that for all but less than κ many $\alpha < \kappa$ such a non-principal ultrafilter exists. Suppose for contradiction that this is not the case. Then there is a set $A \subset \kappa$ of cardinality κ such that $\phi^{-1}\{\phi(\alpha)\}$ contains no non-principal ultrafilters for all $\alpha \in A$. This means that $\phi^{-1}\{\phi(\alpha)\}$ consists only of isolated points of $\beta\kappa$, but it is also a compact set by continuity. Hence each set $\phi^{-1}\{\phi(\alpha)\}$ is finite, for $\alpha \in A$. This implies that $\{\phi(\alpha) : \alpha \in A\}$ has cardinality κ. Now we prove that each point $\phi(\alpha)$ is an isolated point of the range $\phi(\beta\kappa) \subset B_{Z^*}$. This is a contradiction with the fact that B_{Z^*} has weight less than κ since Z had density less than κ. So suppose that $\phi(\alpha)$ is not isolated in that range. Since κ is dense in $\beta\kappa$, we must have

$$\phi(\alpha) \in \overline{\{\phi(\beta) : \beta < \kappa,\, \phi(\beta) \neq \phi(\alpha)\}}.$$

Consider

$$\mathcal{F} = \big\{B \subset \kappa : \exists W \text{ neighbourhood of } \phi(\alpha) \text{ with } \kappa \cap \phi^{-1}(W \setminus \{\phi(\alpha)\}) \subset B\big\}.$$

This is a filter of subsets of κ that contains all complements of finite sets, and satisfies $\{\phi(\alpha)\} = \bigcap_{B\in\mathcal{F}} \overline{\phi(B)}$. There is a non-principal ultrafilter $\mathcal{U}$ on κ that contains $\mathcal{F}$, and we have

$$\phi(\mathcal{U}) = \phi\Big(\lim_{\mathcal{U}} \beta\Big) = \lim_{\mathcal{U}} \phi(\beta) = \phi(\alpha).$$

This contradicts $\phi^{-1}\{\alpha\}$ containing no non-principal ultrafilter.

Fourth step: Definition of the map $g : \kappa \to B_{X^*}$.

Let $\{X_\gamma : \gamma < \mathrm{cf}(\kappa)\}$ be a family consisting of subspaces of X of density character strictly less than κ such that every subspace of X with density character strictly less than $\mathrm{cf}(\kappa)$ is contained in some X_γ. Using the previous step, for each $\gamma < \mathrm{cf}(\kappa)$, we can inductively choose $\alpha(\gamma) < \kappa$ and $g_\gamma \in B_{X^*}$ such that g_γ vanishes on Y, $T_{\alpha(\gamma)}|_{X_\gamma} = g_{\alpha(\gamma)}|_{X_\gamma}$ and $\alpha(\gamma') \neq \alpha(\gamma)$ for $\gamma' < \gamma$.

It is now legitimate to define

$$g(\alpha) := \begin{cases} g_{\alpha(\gamma)} & \text{if } \alpha = \alpha(\gamma) \text{ for some } \gamma < \mathrm{cf}(\kappa), \\ 0 & \text{if } \alpha \notin \{\alpha(\gamma) : \gamma < \mathrm{cf}(\kappa)\}. \end{cases}$$

Finally, we can conclude the proof of the theorem. For this purpose, fix a subspace $Z \subset X$ with $\mathrm{dens}(Z) < \mathrm{cf}(\kappa)$ and find $\gamma < \mathrm{cf}(\kappa)$ such that $Z \subset X_\gamma$. By construction, $T_{\alpha(\gamma)}(x) = g_{\alpha(\gamma)}(x)$ for all $x \in X_\gamma$. By the second step, we can find an element $y \in S_{(X,|||\cdot|||)}$ such that $|||x + ty||| \leq |||x||| \vee |t|$ for all $x \in X_\gamma$ and $t \in \mathbb{R}$. ∎

As an application of the above results we get the following.

COROLLARY 3.8. *ℓ_∞/c_0 admits an $SQ_{<\mathrm{cf}(\mathfrak{c})}$ equivalent norm, and in particular, an $SQ_{\aleph_0}$ equivalent norm.*

Proof. ℓ_∞/c_0 contains a subspace isometric to $c_0(\mathfrak{c})$, coming from an almost disjoint family of cardinality $\mathfrak{c}$, and $\mathrm{cf}(\mathfrak{c}) > \aleph_0$. ∎

4. Stability results. In this section we aim to produce more examples of Banach spaces which are transfinite (A)SQ, by taking direct sums, tensor products and ultrapowers of Banach spaces.

4.1. Direct sums. It is known that the only possible sums which may preserve ASQ are the c_0- and the ℓ_∞-sums [16, Theorem 3.1]. Because of that, we will only focus on these two cases.

Given a family of Banach spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$, we denote

$$\ell_\infty(\mathcal{A}, X_\alpha) := \Big\{ f\colon \mathcal{A} \to \prod_{\alpha\in\mathcal{A}} X_\alpha : f(\alpha) \in X_\alpha \ \forall\alpha \text{ and } \bigvee_{\alpha\in\mathcal{A}} \|f(\alpha)\| < \infty \Big\}.$$

THEOREM 4.1. *Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of Banach spaces and κ an infinite cardinal. If, for every $\varepsilon > 0$, there exists $\beta \in \mathcal{A}$ such that, for every set $A \in \mathcal{P}_\kappa(S_{X_\beta})$, there is $y \in S_{X_\beta}$ satisfying*

$$\|x + y\| \leq 1 + \varepsilon \quad \text{for all } x \in A,$$

then $\ell_\infty(\mathcal{A}, X_\alpha)$ is $ASQ_{<\kappa}$. Moreover, if $|\mathcal{A}| < \mathrm{cf}(\kappa)$ and $\lambda^{|\mathcal{A}|} < \kappa$ for every cardinal $\lambda < \kappa$, then the converse holds too.

Proof. Fix $A \in \mathcal{P}_{<\kappa}(S_{\ell_\infty(\mathcal{A},X_\alpha)})$ and $\varepsilon > 0$. Find $\beta \in \mathcal{A}$ as in the assumption. Then there exists $y \in S_{X_\beta}$ satisfying

$$\|x(\beta) + y\| \leq (1+\varepsilon)(\|x(\beta)\| \vee 1) = 1 + \varepsilon \quad \text{for all } x \in A.$$

We conclude that

$$\|x + y \cdot e_\beta\|_\infty = \bigvee_{\alpha\in\mathcal{A}\setminus\{\beta\}} \|x(\alpha)\| \vee \|x(\beta) + y\| \leq 1 \vee (1+\varepsilon) = 1 + \varepsilon$$

for every $x \in A$. Hence $\ell_\infty(\mathcal{A}, X_\alpha)$ is $\mathrm{ASQ}_{<\kappa}$.

For the "moreover" part, suppose that $\ell_\infty(\mathcal{A}, X_\alpha)$ is $\mathrm{ASQ}_{<\kappa}$ and, for contradiction, that there exists $\varepsilon > 0$ such that for every $\alpha \in \mathcal{A}$ there exists a set $A_\alpha \in \mathcal{P}_{<\kappa}(S_{X_\alpha})$ such that for every $y \in S_{X_\alpha}$ there is $x \in A_\alpha$ satisfying

$$\text{either} \quad \|x+y\| > 1+\varepsilon \quad \text{or} \quad \|x-y\| > 1+\varepsilon.$$

Notice that

$$\Big|\prod_{\alpha\in\mathcal{A}} A_\alpha\Big| \le \Big(\sup_{\alpha\in\mathcal{A}} |A_\alpha|\Big)^{|\mathcal{A}|} < \kappa,$$

where the last inequality follows from observing that $\sup_{\alpha\in\mathcal{A}} |A_\alpha| < \kappa$ since $|\mathcal{A}| < \mathrm{cf}(\kappa)$ and $\lambda^{|\mathcal{A}|} < \kappa$ by hypothesis for every cardinal $\lambda < \kappa$. Since $\ell_\infty(\mathcal{A}, X_\alpha)$ is $\mathrm{ASQ}_{<\kappa}$, we can find $y \in S_{\ell_\infty(X_\alpha)}$ such that

$$\|x+y\|_\infty \le 1+\varepsilon/2 \quad \text{for every } x \in \prod_{\alpha\in\mathcal{A}} A_\alpha.$$

Find $\beta \in \mathcal{A}$ with $\|y(\beta)\| \ge 1-\varepsilon/2$. Then, for every $x \in \prod_{\alpha\in\mathcal{A}} A_\alpha$, we get

$$\begin{aligned} 1+\varepsilon/2 &\ge \|x+y\|_\infty \ge \|x(\beta)+y(\beta)\| \\ &\ge \Big\|x(\beta)+\frac{y(\beta)}{\|y(\beta)\|}\Big\| - \Big\|y(\beta)-\frac{y(\beta)}{\|y(\beta)\|}\Big\| \ge \Big\|x(\beta)+\frac{y(\beta)}{\|y(\beta)\|}\Big\| - \varepsilon/2. \end{aligned}$$

This implies that $\big\|x(\beta)+y(\beta)/\|y(\beta)\|\big\| \le 1+\varepsilon$, which is a clear contradiction since this holds for every $x(\beta) \in A_\beta$. ∎

The following consequence of Theorem 4.1 was used in the proof of Proposition 3.5.

Corollary 4.2. *Let X and Y be Banach spaces and κ an infinite cardinal. Then $X \oplus_\infty Y$ is (A)$SQ_{<\kappa}$ if, and only if, either X or Y is (A)$SQ_{<\kappa}$.*

Proof. Notice that, with the notation from the statement of Theorem 4.1, $|\mathcal{A}| = 2 < \mathrm{cf}(\kappa)$ and that $\lambda^{|A|} = \lambda\cdot\lambda = \lambda < \kappa$ for every $\lambda < \kappa$. Therefore we can apply both directions of Theorem 4.1, hence the ASQ case follows. For the SQ case, notice that the first half of the proof of Theorem 4.1 holds also if we choose $\varepsilon = 0$. Moreover, the second half of the proof of Theorem 4.1 holds for $\varepsilon = 0$ too if $\mathcal{A}$ is finite. ∎

Given a family of Banach spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$, we denote

$$c_0(\mathcal{A}, X_\alpha) := \big\{f \in \ell_\infty(\mathcal{A}, X_\alpha) : (\forall\varepsilon > 0)\ |\{\alpha \in \mathcal{A} : |f(\alpha)| \ge \varepsilon\}| \text{ is finite}\big\}.$$

It is known that, given any sequence $\{X_n : n \in \mathbb{N}\}$ of Banach spaces, the Banach space $c_0(\mathbb{N}, X_n)$ is always ASQ [2, Example 3.1]. A transfinite generalisation of this result is the following.

Proposition 4.3. *Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be an uncountable family of Banach spaces. Then the Banach space $c_0(\mathcal{A}, X_\alpha)$ is $SQ_{<|\mathcal{A}|}$.*

Proof. Fix $A \in \mathcal{P}_{<|\mathcal{A}|}(S_{c_0(\mathcal{A},X_\alpha)})$. For every $x \in A$, $\operatorname{supp}(x)$ is at most countable; therefore, since $\mathcal{A}$ is uncountable,

$$\Big|\bigcup_{x\in A} \operatorname{supp}(x)\Big| \leq |A| \cdot \aleph_0 < |\mathcal{A}|.$$

Find some $\beta \in \mathcal{A} \setminus \bigcup_{x\in A} \operatorname{supp}(x)$ and notice that $\|x + e_\beta\| = 1$ for every $x \in A$. ∎

4.2. Tensor products. In this subsection we give examples of projective and injective tensor products of Banach spaces which are transfinite (A)SQ. Our motivation for this is the known stability results of regular ASQ by taking tensor products coming from [21, 26]. By doing this, we are enlarging the class of (A)SQ spaces.

Let us begin with the projective tensor product, which we briefly recall. Recall that, given two Banach spaces X and Y, the *projective tensor product* of X and Y, denoted by $X \widehat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\| := \inf \Big\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \Big\}.$$

It is known that $B_{X\widehat{\otimes}_\pi Y} = \overline{\operatorname{co}}(B_X \otimes B_Y) = \overline{\operatorname{co}}(S_X \otimes S_Y)$ [27, Proposition 2.2]. Moreover, it is well known that $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ (see [27] for background on tensor products).

In [26, Theorem 2.1], it was proved that, if X and Y are ASQ, then $X \widehat{\otimes}_\pi Y$ is ASQ. The proof is based on averaging techniques in Banach spaces. In the following result we will obtain a transfinite version, which will give us more examples of transfinite (A)SQ spaces.

Let us stress that both X and Y are assumed to be (A)SQ$_{<\kappa}$.

Proposition 4.4. *Let κ be an uncountable cardinal. If X and Y are (A)SQ$_{<\kappa}$, then $X \widehat{\otimes}_\pi Y$ is (A)SQ$_{<\kappa}$.*

Proof. We prove the ASQ case only, the other is similar. To this end, let $A \in \mathcal{P}_{<\kappa}(S_{X\widehat{\otimes}_\pi Y})$ and $\varepsilon > 0$. Since $S_{X\widehat{\otimes}_\pi Y} = \overline{\operatorname{co}}(S_X \otimes S_Y)$, for every $u \in A$ and $n \in \mathbb{N}$ we can find $m_n \in \mathbb{N}$, $\lambda_i^{u,n} \geq 0$, $x_i^{u,n} \in S_X$ and $y_i^{u,n} \in S_Y$ for $i \in \{1, \dots, m_n\}$ such that

$$\Big\| u - \sum_{i=1}^{m_n} \lambda_i^{u,n} x_i^{u,n} \otimes y_i^{u,n} \Big\| \leq 1/n \quad \text{and} \quad \sum_{i=1}^{m_n} \lambda_i^{u,n} = 1.$$

Since κ is uncountable, the sets $\{x_i^{u,n} : u \in A,\, n \in \mathbb{N} \text{ and } i \in \{1, \dots, m_n\}\}$ and $\{y_i^{u,n} : u \in A,\, n \in \mathbb{N} \text{ and } i \in \{1, \dots, m_n\}\}$ have cardinality $< \kappa$, therefore we can find $x \in S_X$ and $y \in S_Y$ satisfying

$$\|x_i^{u,n} + x\| \leq (1 + \varepsilon)^{1/2} \quad \text{and} \quad \|y_i^{u,n} + y\| \leq (1 + \varepsilon)^{1/2}$$

for all $u \in A$, $n \in \mathbb{N}$ and $i \in \{1, \ldots, m_n\}$. Thanks to [26, Lemma 2.2], $\|x_i^{u,n} \otimes y_i^{u,n} + x \otimes y\| \le 1 + \varepsilon$ for every $u \in A$, $n \in \mathbb{N}$ and $i \in \{1, \ldots, m_n\}$. It is clear that

$$\begin{aligned} \|u + x \otimes y\| &\le \Big\| \sum_{i=1}^{m_n} \lambda_i^{u,n} (x_i^{u,n} \otimes y_i^{u,n} + x \otimes y) \Big\| + 1/n \\ &\le \sum_{i=1}^{m_n} \lambda_i^{u,n} \|x_i^{u,n} \otimes y_i^{u,n} + x \otimes y\| + 1/n \\ &\le (1+\varepsilon) \sum_{i=1}^{m_n} \lambda_i^{u,n} + 1/n = 1 + \varepsilon + 1/n \end{aligned}$$

for every $u \in A$ and $n \in \mathbb{N}$. In other words, $\|x \otimes y + u\| \le 1 + \varepsilon$ for every $u \in A$, and the proof is finished. ∎

Remark 4.5. In general, we cannot prove that a projective tensor product $X \widehat{\otimes}_\pi Y$ is $\mathrm{ASQ}_{<\kappa}$ if we only require one factor to be $\mathrm{ASQ}_{<\kappa}$. Indeed, if we take $X = c_0(\kappa)$ and $Y = \ell_p$ for $2 < p < \infty$, we find, from [22, Theorem 3.8], that $X \widehat{\otimes}_\pi Y$ fails to be ASQ (it even contains a convex combination of slices of diameter smaller than 2).

Now we turn our attention to when a space of operators can be transfinite ASQ, a study that will cover the injective tensor product too. Let X and Y be Banach spaces. Given an infinite cardinal κ, denote

$$L_\kappa(Y, X) := \{T \in L(Y, X) : \mathrm{dens}(T(Y)) \le \kappa\}.$$

Using the ideas in [21, Theorem 2.6], we get the following.

Proposition 4.6. *Let $\lambda < \kappa$ be infinite cardinals, and X and Y be non-trivial Banach spaces. Suppose that X is (A)$SQ_{<\kappa}$.*

(a) *If $H \subset L_\lambda(Y, X)$ is a closed subspace such that $Y^* \otimes X \subset H$, then H is (A)$SQ_{<\kappa}$.*
(b) *If $H \subset L_\lambda(Y^*, X)$ is a closed subspace such that $Y \otimes X \subset H$, then H is (A)$SQ_{<\kappa}$.*

Proof. We prove only (a) in the ASQ case. Fix $\mathcal{T} \in \mathcal{P}_{<\kappa}(S_H)$ and $\varepsilon > 0$. Consider the subspace

$$\mathcal{T}(Y) := \bigcup_{T \in \mathcal{T}} T(Y)$$

and notice that $\mathrm{dens}(\mathcal{T}(Y)) \le |\mathcal{T}| \cdot \lambda < \kappa$. By assumption, there exists $x \in S_X$ satisfying

$$\|z + rx\| \le (1+\varepsilon)(\|z\| \vee |r|) \quad \text{for every } z \in \mathcal{T}(Y) \text{ and } r \in \mathbb{R}.$$

Fix any $y^* \in S_{Y^*}$. Then the element $y^* \otimes x \in S_H$ satisfies, for every $T \in \mathcal{T}$

and $y \in S_Y$,

$$\|(T + y^* \otimes x)(y)\| = \|T(y) + y^*(y) \cdot x\| \leq (1+\varepsilon)(\|T(y)\| \vee |y^*(y)|) \leq 1 + \varepsilon.$$

If we pass to the sup on the left-hand side, we conclude that $\|T + y^* \otimes x\| \leq 1 + \varepsilon$ for every $T \in \mathcal{T}$, as desired. ■

Recall that, given two Banach spaces X and Y, the *injective tensor product of X and Y*, denoted by $X \widehat{\otimes}_\varepsilon Y$, is the closure (in the operator norm topology) of the space of finite-rank operators from Y^* to X. Taking this into account, the following corollary is clear from Proposition 4.6. This should be compared with [21, Corollary 2.8].

COROLLARY 4.7. *Let κ be an uncountable cardinal, and X and Y be non-trivial Banach spaces. If X is (A)$SQ_{<\kappa}$, then $X \widehat{\otimes}_\varepsilon Y$ is (A)$SQ_{<\kappa}$.*

4.3. Ultrapowers. In this subsection we will provide examples of ultrapowers of Banach spaces which are transfinite ASQ. Our motivation comes from [17], where it is proved that, in our language, the ultrapower of a Banach space X is $\mathrm{SQ}_{<\aleph_0}$ if, and only if, X is ASQ.

Let us start with a bit of notation. Given a family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of Banach spaces for an infinite set $\mathcal{A}$, and given a non-principal ultrafilter $\mathcal{U}$ over $\mathcal{A}$, consider $c_{0,\mathcal{U}}(\mathcal{A}, X_\alpha) := \{f \in \ell_\infty(\mathcal{A}, X_\alpha) : \lim_{\mathcal{U}} \|f(\alpha)\| = 0\}$. The *ultrapower of $\{X_\alpha : \alpha \in \mathcal{A}\}$ with respect to $\mathcal{U}$* is the Banach space

$$(X_\alpha)_{\mathcal{U}} := \ell_\infty(\mathcal{A}, X_\alpha)/c_{0,\mathcal{U}}(\mathcal{A}, X_\alpha).$$

We will naturally identify a bounded function $f\colon \mathcal{A} \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ with the element $(f(\alpha))_{\alpha\in\mathcal{A}}$. In this way, we denote by $(x_\alpha)_{\alpha,\mathcal{U}}$ or simply by $(x_\alpha)_{\mathcal{U}}$, if no confusion is possible, the coset in $(X_\alpha)_{\mathcal{U}}$ given by $(x_\alpha)_{\alpha\in\mathcal{A}} + c_{0,\mathcal{U}}(\mathcal{A}, (X_\alpha))$.

From the definition of the quotient norm, it is not difficult to prove that $\|(x_\alpha)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_\alpha\|$ for every $(x_\alpha)_{\mathcal{U}} \in (X_\alpha)_{\mathcal{U}}$. We refer the reader to [19] for background about ultraproducts.

Now we are ready to prove the following result.

PROPOSITION 4.8. *Let $\mathcal{A}$ be an infinite set and $\{X_\alpha : \alpha \in \mathcal{A}\}$ a family of $ASQ_{<\kappa}$ spaces. If $\mathcal{U}$ is an $\aleph_1$-incomplete non-principal ultrafilter over $\mathcal{A}$, then $(X_\alpha)_{\mathcal{U}}$ is $SQ_{<\kappa}$.*

Proof. Since $\mathcal{U}$ is $\aleph_1$-incomplete, we can find a function $f : \mathcal{A} \to \mathbb{R}$ such that $f(\alpha) > 0$ for every $\alpha \in \mathcal{A}$ and so that $\lim_{\mathcal{U}} f(\alpha) = 0$.

Let us now prove that $(X_\alpha)_{\mathcal{U}}$ is $\mathrm{SQ}_{<\kappa}$. To this end, fix a set $A \in \mathcal{P}_{<\kappa}(S_{(X_\alpha)_{\mathcal{U}}})$. Without loss of generality we can assume that $\|x(\alpha)\| = 1$ for every $\alpha \in \mathcal{A}$ and every $x \in A$. Now, since X_α is $\mathrm{ASQ}_{<\kappa}$, we can find, for every $\alpha \in \mathcal{A}$, an element $y_\alpha \in S_{X_\alpha}$ such that

$$\|x(\alpha) + y_\alpha\| \leq 1 + f(\alpha) \quad \text{for every } x \in A.$$

Now consider $(y_\alpha)_\mathcal{U} \in S_{(X_\alpha)_\mathcal{U}}$; we prove that it satisfies the desired inequality. To this end fix $x \in A$ and notice that

$$\|x + (y_\alpha)_\mathcal{U}\| = \lim_\mathcal{U} \|x(\alpha) + y_\alpha\| \leq \lim_\mathcal{U}(1 + f(\alpha)) = 1,$$

as required. ■

It is natural, in view of what happens with the behaviour of ASQ in ultrapowers, to ask whether $(X_\alpha)_\mathcal{U}$ ASQ implies X_α ASQ for some $\alpha \in \mathcal{A}$. The following example shows that the answer is no.

EXAMPLE 4.9. Let κ be an infinite cardinal and set $X_n := \ell_n(\kappa)$, where $n \in \mathbb{N}_{\geq 2}$. Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathbb{N}$ and consider $X := (X_n)_\mathcal{U}$. Let us prove that X is $\mathrm{SQ}_{<\kappa}$ in spite of X_n being reflexive for every $n \in \mathbb{N}_{\geq 2}$. Fix $A \in \mathcal{P}_{<\kappa}(S_X)$ and assume that $x(n)$ has norm 1 for each $x \in A$ and $n \in \mathbb{N}_{\geq 2}$.

By the same argument as in Example 3.1 we get elements $y_n \in S_{X_n}$ such that $\|x(n) + y_n\| \leq 2^{1/n}$ for every $x \in A$ and $n \in \mathbb{N}_{\geq 2}$. It is not difficult to show, as before, that

$$\|x + (y_n)_\mathcal{U}\| = \lim_\mathcal{U} \|x(n) + y_n\| \leq \lim_\mathcal{U} 2^{1/n} = 1 \quad \text{for every } x \in A,$$

and the proof is finished.

5. Connections with other properties. It is known that almost square Banach spaces have deep connections with other properties of the geometry of Banach spaces such as diameter 2 properties, octahedrality and the intersection property (see [2]). The aim of the present section is to derive similar connections with transfinite counterparts of the above-mentioned properties.

We will be specially interested in the connection between transfinite versions of almost squareness and octahedrality, because, as a consequence of our work, we will solve an open question from [9]. In order to do so, let us start with the following definition from [9].

DEFINITION 5.1 (see [9, Definitions 2.3 and 5.3]). Let X be a Banach space and κ an uncountable cardinal.

(a) We say that X is *$<\kappa$-octahedral* if, for every subspace $Y \subset X$ with $\mathrm{dens}(Y) < \kappa$ and $\varepsilon > 0$, there exists $x \in S_X$ such that for all $r \in \mathbb{R}$ and $y \in Y$ we have $\|y + rx\| \geq (1 - \varepsilon)(\|y\| + |r|)$.
(b) We say that X fails the *(-1)-$BCP_{<\kappa}$* if, for every subspace $Y \subset X$ with $\mathrm{dens}(Y) < \kappa$, there exists $x \in S_X$ such that for all $r \in \mathbb{R}$ and $y \in Y$ we have $\|y + rx\| = \|y\| + |r|$.

If $\kappa = \aleph_0$, analogous properties are defined by considering finite-dimensional subspaces $Y \subset X$ instead.

It is known that if a Banach space X is ASQ, then X^* is octahedral [2, Proposition 2.5]. In order to solve the above-mentioned open question, our aim will be to establish a transfinite version of this result. We will perform this proof, however, from a more general principle using transfinite versions of the strong diameter 2 property.

DEFINITION 5.2 (see [10, Definitions 2.11 and 2.12]). Let X be a Banach space and κ an infinite cardinal.

(a) We say that X has the *SD2P*$_{<\kappa}$ if, for every $A \in \mathcal{P}_{<\kappa}(S_{X^*})$ and $\varepsilon > 0$, there exist $B \subset S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) \geq 1 - \varepsilon$ for all $x \in B$ and B $(1-\varepsilon)$-*norms* A, that is,
$$\bigvee_{x\in B} y^*(x) \geq 1-\varepsilon \quad \text{for every } y^* \in A.$$
(b) We say that X has the *1-ASD2P*$_{<\kappa}$ if, for every $A \in \mathcal{P}_{<\kappa}(S_{X^*})$, there exist $B \subset S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ for all $x \in B$ and B is norming for A, that is, B 1-norms A.

Recall that, for any infinite cardinal κ, a Banach space X has the SD2P$_{<\kappa}$ if, and only if, X^* is $<\kappa$-octahedral, and that if X has the 1-ASD2P$_{<\kappa}$, then X^* fails the (-1)-BCP$_{<\kappa}$ [10, Theorem 3.2 and Proposition 3.6].

It turns out that if a Banach space is transfinite almost square, then it actually satisfies a transfinite version of the symmetric strong diameter 2 property [3, 16].

DEFINITION 5.3. Let X be a Banach space and κ an infinite cardinal.

(a) We say that X has the *SSD2P*$_{<\kappa}$ if, for every $A \in \mathcal{P}_{<\kappa}(S_{X^*})$ and $\varepsilon > 0$, there exist $B \subset S_X$ and $y \in S_X$ such that B $(1-\varepsilon)$-norms A and $y \pm B \subset (1+\varepsilon)B_X$.
(b) We say that X has the *1-ASSD2P*$_{<\kappa}$ if, for every $A \in \mathcal{P}_{<\kappa}(S_{X^*})$, there exist $B \subset S_X$ and $y \in S_X$ such that B 1-norms A and $y \pm B \subset B_X$.

From the definitions, it is clear that a Banach space has the SSD2P$_{<\kappa}$ (respectively, 1-ASSD2P$_{<\kappa}$) whenever it is ASQ$_{<\kappa}$ (respectively, SQ$_{<\kappa}$).

PROPOSITION 5.4. *Let X be a Banach space and κ an infinite cardinal. If X has the SSD2P$_{<\kappa}$, then X has the SD2P$_{<\kappa}$. Moreover, if κ is uncountable and X has the 1-ASSD2P$_{<\kappa}$, then X has the 1-ASD2P$_{<\kappa}$.*

Proof. We begin by proving that X has the SD2P$_{<\kappa}$. For this purpose, fix $A \in \mathcal{P}_{<\kappa}(S_{X^*})$ and $\varepsilon > 0$. Find $B \subset S_X$ and $y \in S_X$ such that B $(1-\varepsilon/3)$-norms A and $y \pm B \subset (1+\varepsilon/3)B_X$.

We claim that $y + B$ also $(1-\varepsilon)$-norms A. In fact, for every $x^* \in A$ we can find $x \in B$ such that $x^*(x) \geq 1 - \varepsilon/3$ and therefore

$$1 = \|x^*\| \geq \frac{x^*(x \pm y)}{1+\varepsilon/3} \geq \frac{1-\varepsilon/3 \pm x^*(y)}{1+\varepsilon/3},$$

hence $|x^*(y)| \leq 2\varepsilon/3$. We conclude that $x^*(x+y) \geq 1-\varepsilon$, and so the claim is proved.

In order to conclude, we need to find $x^* \in S_{X^*}$ such that $x^*(x+y) \geq 1-\varepsilon$ for every $x \in A$. Any $x^* \in S_{X^*}$ that attains its norm at y satisfies the desired condition: in fact, for every $x \in A$, we have

$$1 = \|x^*\| \geq \frac{x^*(y \pm x)}{1+\varepsilon/3} = \frac{1 \pm x^*(x)}{1+\varepsilon/3},$$

hence $|x^*(x)| \leq \varepsilon/3$ and therefore $x^*(x+y) \geq 1-\varepsilon/3$.

The "moreover" part follows by repeating the same proof with $\varepsilon = 0$ and taking into account that, given any element $x^* \in S_{X^*}$, the set $\{x^*\}$ can be normed using a countable set. ∎

Remark 5.5. In [9, Question 1] it was asked whether every $<\kappa$-octahedral Banach space must contain an isomorphic copy of $\ell_1(\kappa)$. Observe that, by Proposition 5.4, the dual of the space exhibited in Example 3.1 provides a negative answer for every infinite cardinal κ.

6. Parametric ASQ spaces. In this last section we study a further generalisation of ASQ spaces.

Definition 6.1. Let X be a Banach space, κ be a cardinal, and $r, s \in (0,1]$. We say that X is *(r,s)-$SQ_{<\kappa}$* if, for every set $A \in \mathcal{P}_{<\kappa}(S_X)$, there exists $y \in S_X$ satisfying

$$\|rx \pm sy\| \leq 1 \quad \text{for every } x \in A.$$

We say that X is *$(<r,s)$-$SQ_{<\kappa}$* if it is (t,s)-$\mathrm{SQ}_{<\kappa}$ for all $t \in (0,r)$, and similar meaning is given to being *$(r,<s)$-$SQ_{<\kappa}$*. As before, we put "κ" instead of "$<\kappa$" in the definitions above to mean non-strict inequality on the cardinals.

It is clear that being $\mathrm{ASQ}_{<\kappa}$ coincides with being $(<1,<1)$-$\mathrm{SQ}_{<\kappa}$, and that being $\mathrm{SQ}_{<\kappa}$ corresponds to being $(1,1)$-$\mathrm{SQ}_{<\kappa}$.

Remark 6.2. The quantitative version of almost squareness studied in [24], named s-ASQ, corresponds to the space being $(<1,<s)$-$\mathrm{SQ}_{<\aleph_0}$, where $s \in (0,1]$.

Before proceeding, let us prove that every (r,s)-$\mathrm{SQ}_{<\kappa}$ space can be described through subspaces of density character $<\kappa$.

Lemma 6.3. *Let X be a Banach space, $x, y \in S_X$ and $r, s \in (0,1]$. If $\|rx+sy\| \leq 1$, then $\|r'x+s'y\| \leq 1$ for every $r' \in (0,r]$ and $s' \in (0,s]$.*

Proof. Suppose without loss of generality that $s/s' \leq r/r'$ and notice first that

$$\|r'x + sy\| = \frac{r'}{r}\left\|rx + \frac{rs}{r'}y\right\| \leq \frac{r'}{r}\left(\|rx + sy\| + s\left(\frac{r}{r'} - 1\right)\right)$$
$$\leq \frac{r'}{r} + s\left(1 - \frac{r'}{r}\right) \leq 1.$$

We have just proved that $\|r'x + sy\| \leq 1$ for every $r' \in (0, r]$. We can now conclude the proof since

$$\|r'x + s'y\| = \frac{s'}{s}\left\|\frac{r's}{s'}x + sy\right\| \leq \frac{s'}{s} \leq 1,$$

where we have used the first part of the proof together with the fact that $r's/s' \leq r$. ∎

Notice that, thanks to Lemma 6.3, we can conclude that (r, s)-$\mathrm{SQ}_{<\kappa}$ implies (r', s')-$\mathrm{SQ}_{<\kappa}$ whenever $r' \leq r$ and $s' \leq s$.

THEOREM 6.4. *Let X be a Banach space, κ an uncountable cardinal and $r, s \in (0, 1]$. Then X is (r, s)-$SQ_{<\kappa}$ if, and only if, for every subspace $Y \subset X$ with $\mathrm{dens}(Y) < \kappa$, there exists $x \in S_X$ satisfying*

$$\|ry + stx\| \leq \|y\| \vee |t| \quad \text{for all } y \in Y \text{ and } t \in \mathbb{R}.$$

Proof. One implication is obvious. For the converse, we only need to prove the claim when $t \geq 0$. For this purpose, fix a subspace $Y \subset X$ with $\mathrm{dens}(Y) < \kappa$ and find $x \in S_X$ such that

$$\|ry + sx\| \leq 1 \quad \text{for every } y \in S_Y;$$

such an x exists by a density argument. Fix $t \geq 0$, $y \in Y$ and notice that, thanks to Lemma 6.3,

$$\|ry + stx\| = (\|y\| \vee t)\left\|\frac{r\|y\|}{\|y\| \vee t}\frac{y}{\|y\|} + \frac{st}{\|y\| \vee t}x\right\| \leq \|y\| \vee t. \ ∎$$

Now, let us point out some geometrical considerations.

LEMMA 6.5. *Let X be a Banach space.*

(a) *If X is $(1, {<}1)$-$SQ_{<\aleph_0}$ (or just $(1, k)$-SQ_1 for some $k \in (0, 1]$), then B_X cannot contain any extreme point.*
(b) *If X is $({<}1, 1)$-$SQ_{<\aleph_0}$ (or just $(k, 1)$-SQ_1 for some $k \in (0, 1]$), then X is not strictly convex.*

Proof. (a) Let $x \in S_X$. By our assumption we can find $y \in B_X \setminus \{0\}$ such that $\|x \pm y\| \leq 1$. Notice that

$$x = \frac{x + y}{2} + \frac{x - y}{2}.$$

Thus x is a middle point of two distinct elements of B_X and cannot be an extreme point.

(b) Let $x \in kB_X \setminus \{0\}$. By our assumption we can find $y \in S_X$ such that $\|x \pm y\| \leq 1$. Observe that

$$1 = \|y\| = \left\| \frac{y+x}{2} + \frac{y-x}{2} \right\|.$$

From this we deduce, by a simple contradiction argument, that $\|x \pm y\| = 1$ and that y is a norm-1 element which is a middle point of two distinct norm-1 elements, thus proving the claim. ∎

Let us state a simple but useful observation about how (r,s)-SQ$_{<\kappa}$ properties pass from a component to the ∞-sum.

Proposition 6.6. *Let X and Y be non-trivial Banach spaces, $r, s \in (0,1]$, and let κ be a cardinal. If X is (r,s)-SQ$_{<\kappa}$, then $X \oplus_\infty Y$ is (r,s)-SQ$_{<\kappa}$.*

Proof. Fix a set $\{x_\gamma \oplus_\infty y_\gamma\}_{\gamma \in \Gamma} \subset S_{X \oplus_\infty Y}$ with $|\Gamma| < \kappa$. Find $z \in S_X$ such that $\|rx_\gamma/\|x_\gamma\| \pm sz\| \leq 1$ for all $x_\gamma \neq 0$. By Lemma 6.3 we have $\|rx_\gamma + sz\| \leq 1$ (even when $x_\gamma = 0$), so that $\|r(x_\gamma \oplus_\infty y_\gamma) + s(z \oplus_\infty 0)\| \leq 1$ for all $\gamma \in \Gamma$. ∎

Wer now give some first easy examples.

Example 6.7. It is easy to check that c_0 is $(1,{<}1)$-SQ$_{<\aleph_0}$. In fact, for all $x_1, \ldots, x_n \in S_{c_0}$ and $\varepsilon > 0$ we can find $m \in \mathbb{N}$ such that $|x_i(m)| \leq \varepsilon$ for $i \in \{1, \ldots, n\}$ and it is clear that $\|x_i + (1-\varepsilon)e_m\| \leq 1$.

Even more is true: given any sequence $\{X_n : n \in \mathbb{N}\}$ of Banach spaces, $c_0(\mathbb{N}, X_n)$ is $(1, <1)$-SQ$_{<\aleph_0}$. On the other hand, it is trivial to verify that c_0 is not SQ$_{<\aleph_0}$ by considering the element $x = \sum_{n=1}^{\infty} n^{-1} e_n \in S_{c_0}$.

Following the same ideas as in [3, Theorem 2.5], the previous argument can be exploited also to prove more generally that *somewhat regular subspaces* of $C_0(X)$ spaces, where X is some non-compact locally compact and Hausdorff space, are $(1,{<}1)$-SQ$_{<\aleph_0}$.

With similar ideas we can slightly improve the renorming result stated in [8, Theorem 2.3].

Theorem 6.8. *A Banach space X contains an isomorphic copy of c_0 if, and only if, it admits an equivalent $(1,{<}1)$-SQ$_{<\aleph_0}$ norm.*

Proof. Assume that X contains a subspace isometric to c_0. Then there is a subspace Z of X^{**} such that $X^{**} = \ell_\infty \oplus Z$. Consider the norm $|||\cdot|||$ on ℓ_∞ described in Example 2.4. Now, consider on X^{**} the equivalent norm $|||\cdot|||'$ such that $(X^{**}, |||\cdot|||') = (\ell_\infty, |||\cdot|||) \oplus_\infty Z$. By Corollary 4.2, $(X^{**}, |||\cdot|||')$ is SQ$_{<\aleph_0}$ because $(\ell_\infty, |||\cdot|||)$ is SQ$_{<\aleph_0}$.

Now let $x_1 = (u_1, z_1), \ldots, x_k = (u_k, z_k) \in S_X$ and $\varepsilon > 0$. Keeping in mind the notation from Example 2.4, find $n \in \mathbb{N}$ such that $1/n < \varepsilon$ and define $y := (1-\varepsilon)e_m \in B_{c_0}$, where $m \in A_n$. Then a similar calculation to [8, proof of Theorem 2.3] shows that $(y, 0) \in (1-\varepsilon)B_X$ and $\|x_i + (y, 0)\| \leq 1$ for every $i \in \{1, \ldots, k\}$. Hence, X is $(1, {<}1)$-$\mathrm{SQ}_{<\aleph_0}$.

For the converse recall that every Banach space with $(1, {<}1)$-$\mathrm{SQ}_{<\aleph_0}$ norm is ASQ and every ASQ space is known to contain c_0 by [2]. ■

If κ is an infinite cardinal and a Banach space X is $(1, {<}1)$-$\mathrm{SQ}_{<\kappa}$, then a simple transfinite induction proves that X contains an isomorphic copy of $c_0(\kappa)$. Thus, in the case when κ is uncountable, the condition $(1, {<}1)$-$\mathrm{SQ}_{<\kappa}$ is different from $\mathrm{ASQ}_{<\kappa}$ due to Example 3.1.

To give more examples, let us first prove a variation of Theorem 4.1 that we will need.

THEOREM 6.9. *Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of Banach spaces and κ an infinite cardinal. If for every $r \in (0,1)$ there are infinitely many $\alpha \in \mathcal{A}$ such that X_α is (r,r)-$SQ_{<\kappa}$, then $\ell_\infty(\mathcal{A}, X_\alpha)$ is $({<}1, 1)$-$SQ_{<\kappa}$.*

Proof. Fix $r \in (0,1)$ and $A \in \mathcal{P}_{<\kappa}(S_{\ell_\infty(\mathcal{A}, X_\alpha)})$. For every $s \in (r, 1)$ we can find $\alpha(s) \in \mathcal{A}$ and $y_s \in S_{X_{\alpha(s)}}$ satisfying

$$\|sx(\alpha(s)) + sy_s\| \leq 1 \quad \text{for all } x \in A.$$

By our hypothesis, we can assume that, if $s \neq s'$, then $\alpha(s) \neq \alpha(s')$. Define $y \in S_{\ell_\infty(\mathcal{A}, X_\alpha)}$ by

$$y(\alpha) := \begin{cases} sy_s & \text{if } \alpha = \alpha(s) \text{ for some } s \in (r,1), \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to Lemma 6.3, we conclude that

$$\|rx + y\|_\infty = 1 \vee \bigvee_{s \in (r,1)} \|rx(\alpha(s)) + sy_s\| \leq 1 \quad \text{for every } x \in A. \ ■$$

Eventually we can present more examples of (r, s)-$\mathrm{SQ}_{<\kappa}$ spaces that will also show that these properties are actually distinct from the regular $(\mathrm{A})\mathrm{SQ}_{<\kappa}$.

EXAMPLE 6.10. There exists a Banach space X which is M-embedded and strictly convex [18, p. 168]. Therefore X is ASQ [2, Corollary 4.3], but it is neither $(1, {<}1)$-$\mathrm{SQ}_{<\aleph_0}$ nor $({<}1, 1)$-$\mathrm{SQ}_{<\aleph_0}$, by Lemma 6.5.

EXAMPLE 6.11. In the proof of Example 3.1, it is shown that the Banach space $\ell_n(\kappa)$ is $(2^{-1/n}, 2^{-1/n})$-$\mathrm{SQ}_{<\kappa}$. Thus, $X := \ell_\infty(\ell_n(\kappa))$ is $({<}1, 1)$-$\mathrm{SQ}_{<\kappa}$, thanks to Theorem 6.9, but it is not $(1, {<}1)$-$\mathrm{SQ}_{<\aleph_0}$, by Lemma 6.5, since it is a dual space.

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